Differential calculus

Limit of a Function

Definition:

- The limit of a function describes the behavior of that function as the input (usually denoted by x) approaches a certain value.
- In simple terms, it's about asking: "What does f(x) get close to as x gets close to some number?"
- Let f(x) be a real-valued function defined on some open interval around a (but not necessarily at a). We say:

$$\lim_{x \to a} f(x) = L$$

- This means that as x gets arbitrarily close to a, f(x) gets arbitrarily close to L.
- This is a fundamental concept because calculus is built on the idea of making things infinitely small — whether that's change in x, time, or another variable.

Formal (ε-δ) Definition:

- Let $f(x)$be defined on an open interval containing a, except possibly at $a$itself. Then we say:

$$\lim_{x \to a} f(x) = L$$

- if for every number $\varepsilon > 0$, there exists a number $\delta > 0$ such that:

$$0 <| x - a |< \delta \Rightarrow | f(x) - L |< \varepsilon$$

Explanation:

- ε(epsilon) represents how close we want to get to the **limit value** L.
- δ(delta) represents how close we need to make $x$to the point $a$in order to achieve that closeness to L.
    - This definition captures the **rigorous behavior** of a function near a point, not just at the point.

Types of Limits:

A. One-sided limits:

- **Left-hand limit**: $\lim_{x \to a^-} f(x)$
- **Right-hand limit**: $\lim_{x \to a^+} f(x)$
    - A limit exists at a point **a** only if both the left-hand and right-hand limits exist and are equal.

B. Infinite limits:

- If $f(x)$increases or decreases without bound as $x$approaches a point, we write:

- $\lim_{x \to a} f(x) = \infty$
- $\lim_{x \to a} f(x) = -\infty$
  - This doesn't mean the **limit exists** in the real number sense, but rather describes the behavior of the function.

C. Limits at infinity:

- Instead of approaching a finite value a, you can look at the behavior of a function as $x \to \infty$or $x \to -\infty$:
  - $\lim_{x \to \infty} f(x)$
  - $\lim_{x \to -\infty} f(x)$
    - This is crucial in understanding **asymptotic behavior** (e.g., horizontal asymptotes).

Important Properties of Limits:

- Assume $\lim_{x \to a} f(x) = L$and $\lim_{x \to a} g(x) = M$:

1. **Sum Rule:**

$$\lim_{x \to a} [f(x) + g(x)] = L + M$$

2. **Difference Rule:**

$$\lim_{x \to a} [f(x) - g(x)] = L - M$$

3. **Product Rule:**

$$\lim_{x \to a} [f(x) \cdot g(x)] = L \cdot M$$

4. **Quotient Rule (provided $M \neq 0$):**

$$\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{L}{M}$$

5. **Constant multiple:**

$$\lim_{x \to a} [c \cdot f(x)] = c \cdot L$$

Indeterminate Forms:

- While evaluating limits, we often encounter forms like:
  - $\frac{0}{0}$
  - $\frac{\infty}{\infty}$
  - $0 \cdot \infty$

- $\infty - \infty$
- $0^0, \infty^0, 1^\infty$

- These are **indeterminate forms**, meaning that further work is required to find the actual limit — typically via **algebraic simplification**, **L'Hôpital's Rule**, or **series expansion**.

Applications of Limits:

1. Defining the derivative:

- The derivative is defined using the limit:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2. Defining continuity:

- A function is continuous at $a$if:

$$\lim_{x \to a} f(x) = f(a)$$

3. Asymptotic analysis:

- Limits help in determining the end behavior of functions and analyzing convergence/divergence.

Differential Calculus

Definition:

- Differential Calculus is a subfield of calculus that deals with the rate at which quantities change.
- It is centered around the concept of a derivative, which represents an instantaneous rate of change of a function with respect to one of its variables.
- If $y = f(x)$, then the derivative $f'(x)$or $\frac{dy}{dx}$is defined as:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- This tells us how much **y** changes for a tiny change in **x**.

Geometrical Interpretation:

- The derivative of a function at a point gives the **slope of the tangent line** to the curve at that point.
  - If $f'(x) > 0$, the function is **increasing** at x
  - If $f'(x) < 0$, the function is **decreasing** at x

- If $f'(x) = 0$, the function has a **horizontal tangent**, which might be a local max/min or point of inflection

Physical Interpretation:

- In physics and engineering, the derivative often represents **velocity**, **acceleration**, **growth rate**, etc.
- If $s(t)$is position with respect to time, then:
  - $\frac{ds}{dt}$= velocity
  - $\frac{d^2s}{dt^2}$= acceleration

Differentiability:

- A function is said to be **differentiable** at a point if its derivative exists at that point.

Important note:

➢ If a function is **differentiable**, it is also **continuous**, but the converse is not always true.

Example:

➢ $f(x) = |x|$is continuous at $x = 0$, but **not differentiable** there.

Common Derivatives:

Function	Derivative
$\boldsymbol{f(x) = c}$	0
$\boldsymbol{f(x) = x}$	1
$\boldsymbol{f(x) = x^n}$	nx^{n-1}
$\boldsymbol{f(x) = \sin x}$	$\cos x$
$\boldsymbol{f(x) = \cos x}$	$-\sin x$
$\boldsymbol{f(x) = e^x}$	e^x
$\boldsymbol{f(x) = \ln x}$	$\frac{1}{x}$
$\boldsymbol{f(x) = \tan x}$	$\sec^2 x$
$\boldsymbol{f(x) = \arcsin x}$	$\frac{1}{\sqrt{1-x^2}}$
$\boldsymbol{f(x) = \arctan x}$	$\frac{1}{1+x^2}$

Rules of Differentiation:

- We'll explore these in the next topic (Differentiation of sum, product, quotient, etc.), but here's a preview:
  - **Sum Rule**
  - **Product Rule**
  - **Quotient Rule**
  - **Chain Rule**

Applications of Differential Calculus:

1. **Finding maxima and minima** (optimization problems)
2. **Curve sketching** (finding intervals of increase/decrease, concavity, inflection points)
3. **Solving physics problems** (e.g., motion, force, energy)
4. **Economics** (e.g., marginal cost, marginal revenue)
5. **Engineering** (e.g., rate of heat change, stress analysis)

Differentiation of a Sum

Concept:

- When differentiating the sum of two or more functions, the derivative distributes over addition (and subtraction).
- If:

$$f(x) = u(x) + v(x)$$

- Then:

$$f'(x) = u'(x) + v'(x)$$

Why this works:

- This is based on the **linearity** of the derivative operator.
- That is, the rate of change of a combined function is the sum of the individual rates of change.

Example:

$$f(x) = x^2 + 3x + \sin\ x$$

- Then:

$$f'(x) = \frac{d}{dx}(x^2) + \frac{d}{dx}(3x) + \frac{d}{dx}(\sin\ x) = 2x + 3 + \cos\ x$$

Differentiation of a Product

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