Differential Equations

Definition of a Differential Equation

- Differential equations are mathematical equations that relate some function with its derivatives.
- They arise naturally in the modeling of physical systems where quantities change over time or space.

What is a Differential Equation?

- A *differential equation* is an equation that involves:
  - An unknown function (typically dependent on one or more variables),
  - Derivatives of that function.

Example:

$$\frac{dy}{dx} = 3x^2$$

- This is a differential equation where the derivative of $y$with respect to $x$is equal to $3x^2$.

Classification Based on Variables:

- **Ordinary Differential Equations (ODEs):** These involve functions of a single independent variable and their derivatives.
  - Example: $\frac{dy}{dx} + y = e^x$
- **Partial Differential Equations (PDEs):** These involve partial derivatives of functions with more than one independent variable.
  - Example: $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$

Solution of a Differential Equation:

- A *solution* to a differential equation is a function (or family of functions) that satisfies the equation when substituted into it.
- **General Solution:** Contains arbitrary constants; represents a family of solutions.
- **Particular Solution:** Obtained from the general solution by assigning specific values to the constants, usually through initial/boundary conditions.

Why are Differential Equations Important?

- Differential equations are used to model systems in physics, biology, engineering, and economics, such as:
  - Population growth
  - Heat conduction

- Harmonic oscillators
- Electrical circuits
- Fluid flow

Order and Degree of a Differential Equation

- The *order* and *degree* are two important characteristics of a differential equation that help in classifying and choosing solution methods.

Order:

- The **order** of a differential equation is the order (i.e., the highest number of derivatives) of the highest derivative present in the equation.

Example 1:

$$\frac{d^3y}{dx^3} + x\frac{dy}{dx} = 0$$

- Here, the highest derivative is $\frac{d^3y}{dx^3}$, so the **order is 3**.

Degree:

- The **degree** of a differential equation (when defined) is the power of the highest derivative, *after the equation is made polynomial in derivatives* (i.e., no fractional powers or radicals involving derivatives).

Example 2:

$$(\frac{d^2y}{dx^2})^2 + \frac{dy}{dx} = \sin(x)$$

- The highest order derivative is $\frac{d^2y}{dx^2}$raised to the power 2.
- So, the **order is 2** and **degree is 2**.

- **Important Note:** If the equation includes a derivative in a non-polynomial form (like square roots or trigonometric functions of derivatives), the degree is **not defined** unless it is converted to polynomial form.

Steps to Find Order and Degree:

1. **Identify the highest order derivative** – that gives the **order**.
2. If necessary, **remove radicals or fractions** of derivatives to express the equation in polynomial form.
3. **Degree** is the exponent of the highest order derivative in this polynomial form.

More Examples:

Equation	Order	Degree
$\frac{dy}{dx} + y = 0$	1	1
$(\frac{d^2y}{dx^2})^3 + y = 0$	2	3
$(\frac{d^3y}{dx^3})^5 + 2(\frac{dy}{dx})^2 = 0$	3	5
$\frac{d^2y}{dx^2} + \ln(\frac{dy}{dx}) = x$	2	Not defined (logarithm is not polynomial)

Variable Separable Differential Equations

Definition:

- A differential equation is called **variable separable** if it can be expressed in the form:

$$\frac{dy}{dx} = f(x)g(y)$$

- Such that all terms involving $y$can be moved to one side, and all terms involving $x$to the other side:

$$\frac{1}{g(y)}\, dy = f(x)\, dx$$

- This separation of variables allows us to **integrate both sides independently**, and find a solution.

Standard Form:

- A variable separable equation must be rearranged into:

$$N(y)\, dy = M(x)\, dx$$

- Then both sides can be integrated:

$$\int N(y)\, dy = \int M(x)\, dx + C$$

- Where $C$is the constant of integration.

Steps to Solve:

1. Start with the given equation (usually in the form $\frac{dy}{dx} = f(x, y)$).
2. Try to factor and rearrange to isolate y's on one side and x's on the other.

3. Integrate both sides.
4. Simplify and solve for $y$if required.

Example:

- Solve:

$$\frac{dy}{dx} = xy$$

Step 1: Separate variables

$$\frac{1}{y}dy = xdx$$

Step 2: Integrate both sides

$$\int \frac{1}{y}dy = \int xdx$$

$$\ln |y| = \frac{x^2}{2} + C$$

Step 3: Solve for y

$$y = \pm e^{\frac{x^2}{2}+C} = Ae^{\frac{x^2}{2}} \text{(where } A = \pm e^C \text{ is a constant)}$$

- This is the general solution.

Remarks:

- Variable separable equations are **first-order**.
- The method is **not applicable** if you cannot isolate the variables.

Homogeneous Differential Equations (First Order)

Definition:

- A **first-order differential equation** of the form:

$$\frac{dy}{dx} = f(x, y)$$

- is **homogeneous** if the function $f(x, y)$is **homogeneous of degree 0**, i.e.,

$$f(kx, ky) = f(x, y) \text{for all } k \in \mathbb{R}$$

- This means that the function $f(x, y)$depends only on the **ratio** y/xor x/y.
- In such cases, the substitution:

$$y = vx \text{or} x = vy$$

- reduces the equation to a separable form.

Standard Approach:

- Given:

$$\frac{dy}{dx} = f(\frac{y}{x})$$

- Substitute:

$$y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

- Then plug back into the original equation and solve for $v$using the **variable separable** method.

Steps to Solve:

1. Identify if $f(x, y)$is homogeneous of degree 0.
2. Use substitution $y = vx$(or $x = vy$if more convenient).
3. Differentiate the substitution to find $\frac{dy}{dx}$in terms of $v$and $\frac{dv}{dx}$.
4. Replace $\frac{dy}{dx}$and $y$in the original equation.
5. Simplify the equation and separate variables to solve.

Example:

- Solve:

$$\frac{dy}{dx} = \frac{x + y}{x}$$

Step 1: Simplify the RHS

$$\frac{dy}{dx} = 1 + \frac{y}{x}$$

➢ This depends only on y/x, so it is homogeneous.

Step 2: Use substitution

➢ Let $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

➢ Substitute into the equation:

$$v + x\frac{dv}{dx} = 1 + v$$

➢ Subtract $v$from both sides:

$$x\frac{dv}{dx} = 1 \Rightarrow \frac{dv}{dx} = \frac{1}{x}$$

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