Laplace Transform

Introduction to Laplace Transform

What is the Laplace Transform?

- The **Laplace Transform** is a powerful integral transform used to convert a time-domain function, typically involving differential equations, into a frequency-domain (or complex-domain) function.
- This makes solving complex differential equations more manageable, especially in systems governed by linear time-invariant (LTI) processes.

It is particularly useful in:

- Engineering (signal processing, control systems)
- Physics (mechanics, electromagnetism)
- Biology (population models)
- Pharmacokinetics and chemical kinetics (drug concentration modeling)

Why Use the Laplace Transform?

- Converts **differential equations** into **algebraic equations**.
- Makes **initial value problems (IVPs)** easier to solve.
- Naturally handles **piecewise** and **discontinuous functions** (e.g., Heaviside step function).
- Useful for systems with **zero or non-zero initial conditions**.

Historical Context:

- The transform is named after **Pierre-Simon Laplace**, a French mathematician who introduced the technique in the 18th century in his work on probability and differential equations.

Definition of the Laplace Transform

- The **Laplace Transform** of a function $f(t)$, defined for $t \geq 0$, is given by:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

- Where:
  - $f(t)$: a real or complex function of time t,
  - s: a complex number, often written as $s = \sigma + i\omega$,
  - $\mathcal{L}\{f(t)\}$: Laplace Transform of $f(t)$,
  - $F(s)$: transformed function in the **frequency (s-) domain**.

Key Conditions:

- The function $f(t)$must be:
  1. **Piecewise continuous** on every finite interval in $[0, \infty)$.
  2. Of **exponential order**: i.e., there exist constants $M > 0$, $a > 0$, such that:

$$| f(t) | \leq Me^{at} \text{for all } t \geq 0$$

- This ensures convergence of the integral.

Interpretation:

- The Laplace Transform can be thought of as a **weighted average** of the function $f(t)$, with the exponential decay e^{-st}assigning higher weight to smaller values of t.
- This allows capturing the behavior of the function as a whole in a new domain.

Properties of the Laplace Transform

These properties are essential for simplifying Laplace transform calculations, solving differential equations, and handling more complex expressions.

A. Linearity Property:

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

- Where $a, b$are constants.

B. First Shifting Theorem (Exponential Shift):

$$\mathcal{L}\{e^{at}f(t)\} = F(s - a)$$

- This shifts the function in the **s-domain**.

C. Second Shifting Theorem (Time Shift):

- If $u_c(t)$is the Heaviside (unit step) function,

$$\mathcal{L}\{f(t - c)u_c(t)\} = e^{-cs}F(s)$$

- Used to model delays or piecewise inputs.

D. Differentiation Property:

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

General form:

$$\mathcal{L}\{\frac{d^n f(t)}{dt^n}\} = s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$$

- This is critical in converting **differential equations into algebraic equations**.

E. Integration Property:

$$\mathcal{L}\{\int_0^t f(\tau)\,d\tau\} = \frac{F(s)}{s}$$

F. Scaling (Time Scaling):

$$\mathcal{L}\{f(at)\} = \frac{1}{a}F(\frac{s}{a})$$

G. Convolution Theorem:

- If $h(t) = (f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$, then:

$$\mathcal{L}\{h(t)\} = F(s)G(s)$$

- Used for **systems response analysis**.

Laplace Transforms of Elementary Functions

- These are standard Laplace transform results for commonly encountered functions.
- Memorizing these is essential for solving differential equations and analyzing systems.

Common Elementary Functions and Their Laplace Transforms

Function $f(t)$	Laplace Transform $\mathcal{L}\{f(t)\} = F(s)$	Conditions
1	$\frac{1}{s}$	$s > 0$
t	$\frac{1}{s^2}$	$s > 0$
t^n	$\frac{n!}{s^{n+1}}$	$s > 0, n \in \mathbb{N} \cup \{0\}$
e^{at}	$\frac{1}{s-a}$	$s > a$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sinh(at)$	$\frac{a}{s^2-a^2}$	(s >
$\cosh(at)$	$\frac{s}{s^2-a^2}$	(s >
Unit Step $u(t-a)$	$\frac{e^{-as}}{s}$	$s > 0$
Dirac Delta $\delta(t-a)$	e^{-as}	$a \geq 0$

Notes:

- These results assume that $f(t) = 0$for $t < 0$(causality condition).

- They form the basis for computing Laplace transforms of more complex functions by using properties such as linearity and shifting.

Linearity Property of Laplace Transform

Statement:

- The Laplace transform is **linear**, meaning the transform of a linear combination of functions is the same linear combination of their transforms.
- Let $f(t)$and $g(t)$be two functions whose Laplace transforms exist, and let $a$and $b$be any constants. Then:

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

Proof:

- Using the definition of Laplace Transform:

$$\mathcal{L}\{af(t) + bg(t)\} = \int_0^\infty e^{-st}[af(t) + bg(t)]dt$$

- By properties of integrals:

$$= a\int_0^\infty e^{-st}f(t)dt + b\int_0^\infty e^{-st}g(t)dt$$

$$= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

- This proves the linearity.

Importance:

- Allows us to decompose complex functions into simpler parts.
- Makes solving linear differential equations systematic.
- Compatible with the principle of superposition (important in physics and engineering).

Example:

- Find the Laplace transform of:

$$f(t) = 3\sin(2t) + 5e^{-t}$$

- Using known transforms:

$$\mathcal{L}\{\sin(2t)\} = \frac{2}{s^2+4}, \mathcal{L}\{e^{-t}\} = \frac{1}{s+1}$$

- Now apply linearity:

$$\mathcal{L}\{f(t)\} = 3\cdot\frac{2}{s^2+4} + 5\cdot\frac{1}{s+1} = \frac{6}{s^2+4} + \frac{5}{s+1}$$

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